FPRAS for computing a lower bound for weighted matching polynomial of graphs

Shmuel Friedland
Department of Mathematics, Statistics, and Computer Science,
University of Illinois at Chicago
Chicago, Illinois 60607-7045, USA

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Abstract

We give a fully polynomial randomized approximation scheme to compute a lower bound for the matching polynomial of any weighted graph at a positive argument. For the matching polynomial of complete bipartite graphs with bounded weights these lower bounds are asymptotically optimal.

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1 Introduction

Let $G = (V, E)$ be an undirected graph, (with no self-loops), on the set of vertices V and the set of edges E. A set of edges $M \subseteq E$ is called a *matching* if no two distinct edges $e_1, e_2 \in M$ have a common vertex. M is called a *k-matching* if $\#M = k$. For $k \in \mathbb{N}$ let $\mathcal{M}_k(G)$ be the set of k-matchings in G. ($\mathcal{M}_k(G) = \emptyset$ for $k > \lfloor \frac{\#V}{2} \rfloor$.) If $\#V = 2n$ is even then an n-matching is called a *perfect matching*.

Let $\omega : E \to (0, \infty)$ be a weight function, which associate with edge $e \in E$ a positive weight $\omega(e)$. We call $G_\omega = (V, E, \omega)$ a weighted graph. Denote by ι the weight $\iota : E \to \{1\}$. Then G can be identified with G_ω.

Let $M \in \mathcal{M}_k(G)$. Then the weight of the matching is defined as $\omega(M) := \prod_{e \in M} \omega(e)$. *The total weighted k-matching* of G_ω is defined:

$$\phi(k, G_\omega) := \sum_{M \in \mathcal{M}_k(G)} \omega(M), k \in \mathbb{N}$$

where $\phi(k, G_\omega) = 0$ if $\mathcal{M}_k(G) = \emptyset$ for any $k \in \mathbb{N}$. Furthermore we let $\phi(0, G_\omega) := 1$. Note that $\phi(k, G_\iota) = \#\mathcal{M}_k(G)$, i.e. the number of k-matchings in G for any $k \in \mathbb{N}$. The *weighted matching polynomial* of G_ω is defined by:

$$\Phi(t, G_\omega) := \sum_{k=0}^{n} \phi(k, G_\omega) t^{n-k}, \quad n = \lfloor \frac{\#V}{2} \rfloor.$$

This polynomial is fundamental in the monomer-dimer model in statistical physics [3, 12], and for $\omega = 1$ in combinatorics. Note that if $\#V$ is even then $\Phi(0, G_\omega)$ is the total weighted perfect matching of G. (Some authors consider the polynomial $t^{\lfloor \frac{\#V}{2} \rfloor} \Phi(t^{-1}, G_\omega)$ instead of $\Phi(t, G_\omega)$.) It is known that nonzero roots of a weighted matching polynomial of G are real and negative [12]. Observe that $\Phi(1, G_\iota)$ the total number monomer-dimer coverings of G.

Let G be a bipartite graph, i.e., $V = V_1 \cup V_2$ and $E \subset V_1 \times V_2$. In the special case of a bipartite graph where $n = \#V_1 = \#V_2$, it is well known that $\phi(n, G)$ is given as $\mathrm{perm}\, B(G)$, the permanent of the incidence matrix $B(G)$ of the bipartite graph G. It was shown by Valiant that the computation of the permanent of a $(0, 1)$ matrix is $\#\mathbf{P}$-complete [17]. Hence, it is believed that the computation of the number of perfect matching in a general bipartite graph satisfying $\#V_1 = \#V_2$ cannot be polynomial.

In a recent paper Jerrum, Sinclair and Vigoda gave a *fully-polynomial randomized approximation scheme* (*fpras*) to compute the permanent of a nonnegative matrix [13]. (See also Barvinok [1] for computing the permanents within a simply exponential factor, and Friedland, Rider and Zeitouni [9] for concentration of permanent estimators for certain large positive matrices.) [13] yields the existence a fpras to compute the total weighted perfect matching in a general bipartite graph satisfying $\#V_1 = \#V_2$. In a recent paper of Levy and the author it was shown that there exists fpras to compute the total weighted k-matchings for any bipartite graph G and any integer $k \in [1, \frac{\#V}{2}]$. In particular, the generating matching polynomial of any bipartite graph G has a fpras. This observation can be used to find a fast computable approximation to the *pressure* function, as discussed in [8], for certain families of infinite graphs appearing in many models of statistical mechanics, like the integer lattice $\mathbb{Z}^d$.

The MCMC, (Monte Carlo Markov Chain), algorithm for computing the total weighted perfect matching in a general bipartite graph satisfying $\#V_1 = \#V_2$, outlined in [13], can be applied to estimate the total weighted perfect matchings in a weighted non-bipartite graph with even number of vertices. However the proof in [13], that shows this algorithm is frpas for bipartite graphs, fails for non-bipartite graphs. Similarly, the proof of concentration results given in [9] do not seem to work for non-bipartite graphs. The technique introduced by Barvinok in [1] to estimate the number of weighted perfect matching in bipartite graphs, does extend to the estimate of total weighted perfect matchings in a general non-bipartite graph with even number of vertices, when one uses real or complex Gaussian distribution. (See the discussion in §5.)

In this paper we give a fpras for computing a lower bound $\tilde\Phi(t, G_\omega)$ for the weighted generated function $\Phi(t, G_\omega)$ for a fixed $t > 0$. We show that this lower bound has a multiplicative error at most $\exp(N \min(\frac{a^2}{2t}, C_1))$, see (1.7), where a^2 is the maximal weight of edges of G and

$$C_1 = -\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \log(x^2) e^{-\frac{x^2}{2}} dx = 1.270362845\ldots \tag{1.1}$$

These estimates are similar in nature to heuristic computations of Baxter [2], where he showed that his computation for the dimers on $\mathbb{Z}^2$ lattice are very precise away from only dimer configurations, i.e. perfect matchings. (The results of heuristic computations of Baxter were recently confirmed in [8].) We show that that for the matching polynomial of complete bipartite graphs with weights in $[b^2, a^2], 0 < b \le a$, this lower bound is asymptotically optimal.

We now describe briefly our technical results. With each weighted graph G_ω associate a skew symmetric matrix $A = [a_{ij}]_{i,j=1}^N \in \mathbb{R}^{N\times N}$, $A^\top = -A$, where $N := \#V$, as follows. Identify E with $\langle N\rangle := \{1, \ldots, N\}$, and each edge $e \in E$ with the corresponding unordered pair $(i,j), i \ne j \in \langle N\rangle$. Then $a_{ij} \ne 0$ if and only $(i,j) \in E$. Furthermore for $1 \le i < j \le N, (i,j) \in E$ $a_{ij} = \sqrt{\omega((i,j))}$. For $1 \le i \le j \le \mathbb{N}$ let x_{ij} be a set of $\binom{N}{2}$ independent random variables with

$$\mathrm{E}\ x_{ij} = 0, \quad \mathrm{E}\ x_{ij}^2 = 1, \quad 1 \le i \le j \le N. \tag{1.2}$$

Let $\mathbf{x} := (x_{11}, \ldots, x_{1N}, x_{22}, \ldots, x_{NN})$. We view $\mathbf{x}$ as a random vector variable with values $\boldsymbol{\xi} = (\xi_{11}, \ldots, \xi_{NN}) \in \mathbb{R}^{\binom{N+1}{2}}$. Let Y_A be the following skew-symmetric random matrix

$$Y_A := [a_{ij} x_{\min(i,j)\max(i,j)}]_{i,j=1}^N \in \mathbb{R}^{N\times N}. \tag{1.3}$$

A variation of the Godsil-Gutman estimator [10] states

$$\mathrm{E}\ \det(\sqrt{t} I_N + Y_A)) = \Phi(t, G_\omega) \text{ if } N = \#V \text{ is even,} \tag{1.4}$$
$$\mathrm{E}\ \det(\sqrt{t} I_N + Y_A) = \sqrt{t}\Phi(t, G_\omega) \text{ if } N = \#V \text{ is odd.} \tag{1.5}$$

for any $t \ge 0$. Here I_N stands for $N \times N$ identity matrix.

We show the concentration of $\log\det(\sqrt{t} I_N + Y_A)$ around

$$\log \tilde\Phi(t, G_\omega) := \mathrm{E}\ \log\det(\sqrt{t} I_N + Y_A) \tag{1.6}$$

using [11]. These concentration results show that $\tilde\Phi(t, G_\omega)$ has a fpras. Jensen inequalities yield that $\tilde\Phi(t, G_\omega) \le \Phi(t, G_\omega)$. Together with an upper estimate we have the following bounds:

$$\frac{1}{N}\log\tilde\Phi(t, G_\omega) \le \frac{1}{N}\log\Phi(t, G_\omega) \le \frac{1}{N}\log\tilde\Phi(t, G_\omega) + \min(\frac{a^2}{2t}, C_1) \tag{1.7}$$

where $a = \max|a_{ij}|$. The above inequality hold also for $t = 0$. (For N even and $t = 0$ this result is due to Barvinok [1, §7].) It is our hope that by refining the techniques we are using one can show that $\Phi(t, G_\omega)$ has fpras for any $t > 0$.

2 Preliminary results

Lemma 2.1 *Let $G = (V, E)$ be an undirected graph on N vertices. Let $\omega : V \to (0, \infty)$ be a given weight function. Let $A = -A^\top \in \mathbb{R}^{n\times n}$ be the corresponding real skew symmetric matrix defined in §1. Assume that $x_{ij}, i = 1, \ldots, j, j = 1, \ldots, N$ are $\binom{N+1}{2}$ independent random variables, normalized by the conditions (1.2). Let $Y_A \in \mathbb{R}^{N\times N}$ be the skew symmetric real matrix defined by (1.3). Then (1.4-1.5) hold.*

Proof. Let $\sqrt{t} = s$. Observe first that $\det(sI_N + Y_A)$ is a sum of $N!$ monomials, where each monomial is of degree at most 2 in the variables x_{ij} for $i < j$ and of degree m invariable s. The total degree of each monomial is N. The expected value of such a monomial is zero if at least the degree of one of the variables x_{ij} is one. So it is left to consider the expected value of all monomials, where the degree if each x_{ij} is 0 or 2, which are called nontrivial monomials.

Assume first that N is even. Observe that if a monomial contains s of odd power than it must be linear at least in one x_{ij}. Hence its expected value is zero. Thus E $\det(sI_N + Y_A)$ is a polynomial in s^2. Consider a nontrivial monomial such that the power of s is $N - 2m$. Note that this monomial is of the form $\tau s^{N-2m} \prod_{(i,j)\in M} \omega((i, j))x_{ij}^2$, for some m matching $M \in \mathcal{M}_m$. Here $(-1)^m\tau$ is the sign of the corresponding permutation $\sigma : \langle N\rangle \to \langle N\rangle$. Since $\sigma(i) = j, \sigma(j) = i$ for any edge $(i, j) \in M$, and $\sigma(i) = i$ for all vertices i which are not covered by M we deduce that $\tau = 1$. Hence the expected value of this monomial is $s^{N-2m} \prod_{e\in M} \omega(e)$. This proves (1.4). The identity (1.5) is shown similarly. $\square$

Recall the following well known result:

Lemma 2.2 *Let $A = -A^\top \in \mathbb{R}^{N\times N}$ be a skew symmetric matrix. Then $B := \imath A$, where $\imath := \sqrt{-1}$, is a hermitian matrix. Arrange the eigenvalues of B in a decreasing order: $\lambda_1(B) \geq \ldots \geq \lambda_N(B)$. Then*

$$\lambda_{N-i+1}(B) = -\lambda_i(B) \text{ for } i = 1, \ldots, N. \tag{2.1}$$

In particular

$$\det(\sqrt{t}I_N + A) = \prod_{i=1}^{N} \sqrt{t + \lambda_i(B)^2}. \tag{2.2}$$

Proof. Clearly, B is hermitian. Hence all the eigenvalues of B are real. Arrange these eigenvalues in a decreasing order. So $-\imath\lambda_j(B), j = 1, \ldots, N$ are the eigenvalues of A. Since A is real valued, the nonzero eigenvalues of A must be in conjugate pairs. Hence equality (2.1) holds. Observe next that if $\lambda_k(A) = -\imath\lambda_k(B) \neq 0$ then

$$(\sqrt{t} + \lambda_k(A))(\sqrt{t} + \lambda_{N-k+1}(A)) = \sqrt{t + \lambda_k(B)^2}\sqrt{t + \lambda_{N-k+1}(B)^2}.$$

As the eigenvalues of $\sqrt{t}I_N + A$ are $\sqrt{t} + \lambda_k(A), k = 1, \dots, N$ we deduce (2.2). $\square$

3 Concentration for Gaussian entries

In this section we assume that each x_{ij} is a normalized real Gaussian variable, i.e satisfying (1.2). Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is called *Lipschitz* function, or *Lipschtzian*, if there exists $L \in [0, \infty)$ such that $\frac{|f(x)-f(y)|}{|x-y|} \le L$ for all $x \ne y \in \mathbb{R}$. The smallest possible L for a Lipschitz function is denoted by $|f|_{\mathcal{L}}$. Let $\mathrm{A_N} \subset \mathbb{R}^{\mathrm{n}\times\mathrm{n}}, \imath\mathrm{A_N} \subset \mathbb{C}^{\mathrm{n}\times\mathrm{n}}$ denote the set of $N \times N$ real skew symmetric matrices, and the set of $N \times N$ hermitian matrices of the form $\imath A, A \in \mathrm{A_N}$. With each $A \in \mathrm{A_N}$ we associate a weighted graph $G_\omega = (V, E, \omega)$, where $V = \langle N \rangle, (i,j) \in V \iff a_{ij} \ne 0, \omega((i,j)) = |a_{ij}|^2$. Denote by $a := \max |a_{ij}|$. To avoid the trivialities we assume that $a > 0$. Note that a^2 is the maximal weight of the edges in G_ω. Let Y_A be the random skew symmetric matrix given by (1.3) and denote by X_A the random hermitian matrix $X_A := \frac{1}{\sqrt{N}}\imath Y_A$.

Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function. As in [11] consider the following $F : \imath\mathrm{A_N} \to \mathbb{R}$ given by the trace formula:

$$F(B) = \mathrm{tr}_N\, f(B) := \frac{1}{N}\sum_{i=1}^{N} f(\lambda_i(B)), \quad B \in \imath\mathrm{A_N}.$$

Denote by $\mathrm{E}\ \mathrm{tr}_N(f(X_A))$ the expected value of the function $\mathrm{tr}_N(f(X_A))$. The concentration result [11, Thm 1.1(b)] states:

$$\Pr(|\,\mathrm{tr}_N(f(X_A)) - \mathrm{E}\ \mathrm{tr}_N(f(X_A))| \ge r) \le 2e^{-\frac{N^2r^2}{8a^2|f|_{\mathcal{L}}^2}} \tag{3.1}$$

(Recall that the normalized Gaussian distribution satisfies the log Sobolev inequality with $c = 1$.) We remark that since the entries of X_A are either zero or pure imaginary one can replace the factor 8 in the inequality (3.1) by the factor 2. See for example the results in [15, 8.5].

Lemma 3.1 *Let $0 \ne A = [a_{ij}] \in \mathrm{A_N}, \mathrm{a} = \max|\mathrm{a_{ij}}|, \mathrm{t} \in (0,\infty)$, $x_{ij}, 1 \le i \le j \le N$ be independent Gaussian satisfying (1.2). Let $Y_A \in \mathrm{A_N}$ be the random skew symmetric matrix given by (1.3). Then*

$$\Pr(|\log\det(\sqrt{t}I_N + Y_A) - \mathrm{E}\ \log\det(\sqrt{t}I_N + Y_A)| \ge Nr) \le 2e^{-\frac{tNr^2}{2a^2}}. \tag{3.2}$$

Proof. Let $f_t(x) := \frac{1}{2}\log(\frac{t}{N} + x^2)$. f_t is differentiable and

$$|(f_t)_{\mathcal{L}}| = \max_{x\in\mathbb{R}} |f_t'(x)| = \frac{\sqrt{N}}{2\sqrt{t}}.$$

Apply (3.1) to f_t. Observe that the right-hand side of (3.1) is equal to the right-hand side of (3.2). Use (2.2) to deduce that

$$N\,\mathrm{tr}_N(f_t(X_A)) = \sum_{i=1}^{N} \log\sqrt{\frac{t}{N} + \lambda_i(X_A)^2} = \sum_{i=1}^{N} \log\sqrt{\frac{t}{N} + \frac{|\lambda_i(Y_A)|^2}{N}}$$

$$= -\frac{1}{2}N\log N + \log\prod_{i=1}^{N}\sqrt{t + |\lambda_i(Y_A)|^2} = -\frac{1}{2}N\log N + \log\det(\sqrt{t}I_N + Y_A).$$

Hence the left-had sides of (3.1) and (3.2) are equivalent. $\square$

The following lemma is well known, e.g. [9, p'1566], and we bring its proof for completeness.

Lemma 3.2 *Let U be a real random variable with a finite expected value $\mathrm{E}\,U$. Then $e^{\mathrm{E}\,U} \le \mathrm{E}\,e^U$. Assume that the following condition hold*

$$\Pr(U - \mathrm{E}\,U \ge r) \le 2e^{-Kr^2} \text{for each } r \in (0,\infty) \text{ and some } K > 0. \tag{3.3}$$

Then

$$e^{\mathrm{E}\,U} \le \mathrm{E}\,e^U \le e^{\mathrm{E}\,U}\left(1 + \frac{2e^{\frac{1}{4K}}}{\sqrt{K\pi}}\right). \tag{3.4}$$

Proof. Since e^u is convex, the inequality $e^{\mathrm{E}\,U} \le \mathrm{E}\,e^U$ follows from Jensen inequality. Let $\mu := \mathrm{E}\,U$ and $F(u) := \Pr(U \le u)$ be the cumulative distribution function of U. We claim that

$$\mathrm{E}\,e^U \le e^{\mu} + \int_{\mu<u} e^u(1 - F(u))du. \tag{3.5}$$

Clearly

$$\mathrm{E}\,e^U = \int_{-\infty}^{\infty} e^u dF(u) = \int_{u\le\mu} e^u dF(u) + \int_{\mu<u} e^u dF(u). \tag{3.6}$$

Since $e^u \le e^{\mu}$ for $u \le \mu$ we deduce that

$$\int_{u\le\mu} e^u dF(u) \le e^{\mu}F(\mu).$$

We now estimate the second integral in the right-hand side of (3.6). Recall that $F(u)$ is an nondecreasing function continuous from the right satisfying $F(+\infty) = 1$. Hence $e^u(F(u) - 1) \le 0$ for all $u \in \mathbb{R}$. For any $R > \mu$ use integration by parts to deduce

$$\int_{\mu<u\le R} e^u dF(u) = e^u(F(u) - 1)|_{\mu}^{R} + \int_{\mu<u\le R} e^u(1 - F(u))du \le$$

$$e^{\mu}(1 - F(\mu)) + \int_{\mu<u} e^s(1 - F(u))du.$$

So

$$\int_{\mu<u} e^u dF(u) \le e^{\mu}(1-F(\mu)) + \int_{\mu<u} e^u(1-F(u))du,$$

and (3.5) holds.

Assume now that (3.3) holds. Thus

$$1-F(u) = \Pr(U>u) \le 2e^{-K(u-\mu)^2} \text{ for any } u>\mu.$$

Hence

$$\int_{\mu<u} e^u(1-F(u))du \le 2\int_{\mu<u} e^{u-K(u-\mu)^2}du \le$$

$$2e^{\mu}\int_{-\infty}^{\infty} e^{-K(u-\mu-\frac{1}{2K})^2+\frac{1}{4K}}du = \frac{2e^{\mu}e^{\frac{1}{4K}}}{\sqrt{K\pi}}.$$

Combine the above inequality with (3.5) to deduce the right-hand side of (3.4). □

Corollary 3.3 *Let the assumptions of Lemma 3.1 hold. Then*

$$\frac{1}{N}\log\tilde{\Phi}(t,G_\omega) \le \frac{1}{N}\log\Phi(t,G_\omega) \le \frac{1}{N}\log\tilde{\Phi}(t,G_\omega) + \frac{1}{N}\log(1+\frac{\sqrt{8N}ae^{\frac{a^2N}{2t}}}{\sqrt{\pi t}}).$$

4 FPRAS for computing $\log\tilde{\Phi}(t,G_\omega)$

Let $B \in \mathbb{R}^{N\times N}$. For $k\in\mathbb{N}$ denote by $\oplus_k B \in \mathbb{R}^{kN\times kN}$ the block diagonal matrix $\mathrm{diag}(\underbrace{B,\dots,B}_{k})$. ($\oplus_k B$ is a direct sum of k copies of B.) Note that if $B\in \mathrm{A_N}$ then $\oplus_k B\in\mathrm{A_{kN}}$. Clearly,

$$\det(sI_{kN}+\oplus_k B) = (\det(sI_N+B))^k \text{ for any } B\in\mathbb{R}^{N\times N} \text{ and } s\in\mathbb{R}. \quad (4.1)$$

Let $A\in\mathrm{A_N}$, and Y_A be the random matrix defined by (1.3). By $Y_A(\boldsymbol{\xi})$ we mean the skew symmetric matrix $[a_{ij}\xi_{\min(i,j)\max(i,j)}]_{i,j=1}^N$, which is a *sampling* of Y_A. Let $x_{ij}, 1\le i\le j\le kN$ be $\binom{kN+1}{2}$ normal Gaussian independent random variables. Consider the random matrix $Y_{\oplus_k A}$. Then a sampling

$$Y_{\oplus_k A}(\boldsymbol{\xi}), \boldsymbol{\xi}\in\mathbb{R}^{\binom{kN+1}{2}} = \mathrm{diag}(Y_A(\boldsymbol{\xi}_1),\dots,Y_A(\boldsymbol{\xi}_k)), \boldsymbol{\xi}_i\in\mathbb{R}^{\binom{N+1}{2}}, i=1,\dots,k$$

is equivalent to k sampling of Y_A.

Theorem 4.1 *Let* $0\ne A=[a_{ij}]\in\mathrm{A_N}, \mathrm{a}=\max|\mathrm{a_{ij}}|, \mathrm{t}\in(0,\infty)$*,* $x_{ij}, 1\le i\le j\le N$ *be independent Gaussian satisfying (1.2). Let* $Y_A\in\mathrm{A_N}$ *be the*

random skew symmetric matrix given by (1.3). Let $Y_A(\boldsymbol{\xi}_1), \ldots, Y_A(\boldsymbol{\xi}_k)$ be k samplings of Y_A. Then

$$\Pr\Big(\Big|\frac{1}{k}\sum_{i=1}^{k} \log\det(\sqrt{t}I_N + Y_A(\boldsymbol{\xi}_i)) - \log\tilde{\Phi}(t, G_\omega)\Big| \geq Nr\Big) \leq 2e^{-\frac{tkNr^2}{2a^2}}. \quad (4.2)$$

In particular the inequality

$$\frac{1}{N}\log\tilde{\Phi}(t, G_\omega) \leq \frac{1}{N}\log\Phi(t, G_\omega) \leq \frac{1}{N}\log\tilde{\Phi}(t, G_\omega) + \frac{a^2}{2t} \quad (4.3)$$

holds.

Hence an approximation of $\tilde{\Phi}(t, G_\omega)$ by $(\prod_{i=1}^{k}\det(\sqrt{t}I_N + Y_A(\boldsymbol{\xi}_i)))^{\frac{1}{k}}$ is a fully-polynomial randomized approximation scheme.

Proof. Use (4.1) to obtain

$$\log\det(\sqrt{t}I_{kN} + Y_{\oplus_k A}(\boldsymbol{\xi})) = \sum_{i=1}^{k} \log\det(\sqrt{t}I_N + Y_A(\boldsymbol{\xi}_i))$$

Hence

$$\mathrm{E}\ \log\det(\sqrt{t}I_{kN} + Y_{\oplus_k A}) = k\mathrm{E}\ \log\det((\sqrt{t}I_N + Y_A) = k\log\tilde{\Phi}(t, G_\omega) \quad (4.4)$$

Apply (3.2) to $Y_{\oplus_k A}$ to deduce (4.2). Observe next that

$$\mathrm{E}\ \det(\sqrt{t}I_{kN} + Y_{\oplus_k A}) = \mathrm{E}\ \det((\sqrt{t}I_N + Y_A)^k = \Phi(t, G_\omega)^k. \quad (4.5)$$

Use Lemma 3.2 for the random variable $\log\det(\sqrt{t}I_{kN} + Y_{\oplus_k A})$ to deduce

$$\frac{1}{N}\log\tilde{\Phi}(t, G_\omega) \leq \frac{1}{N}\log\Phi(t, G_\omega) \leq \frac{1}{N}\log\tilde{\Phi}(t, G_\omega) + \\ + \frac{1}{kN}\log\Big(1 + \frac{\sqrt{8kN}ae^{\frac{a^2kN}{2t}}}{\sqrt{\pi t}}\Big).$$

Let $k \to \infty$ to deduce (4.3).

We now show that (4.2) gives fpras for computing $\tilde{\Phi}(t, G_\omega)$ in sense of [14]. Let $\epsilon, \delta \in (0, 1)$. Choose

$$r = \frac{\epsilon}{2N}, \quad k = \Big\lceil\frac{8a^2N\log\frac{4}{\delta}}{t\epsilon^2}\Big\rceil.$$

Then

$$\Pr\Big(1 - \epsilon < \frac{(\prod_{i=1}^{k}\det(\sqrt{t}I_N + Y_A(\boldsymbol{\xi}_i)))^{\frac{1}{k}}}{\tilde{\Phi}(t, G_\omega)} < 1 + \epsilon\Big) > 1 - \frac{\delta}{2}.$$

Observe next that

$$\Pr(|x_{ij}| > \sqrt{2\log\frac{N^2k}{\delta}}\ ) < \frac{\delta}{N^2k}.$$

Hence with probability $1-\frac{\delta}{2}$ at least, the absolute of each off-diagonal of $Y_A(\boldsymbol{\xi}_i)), i = 1,\ldots,k$ is bounded by $a\sqrt{2\log\frac{N^2k}{\delta}}$. In this case all the entries of $\sqrt{t}I_N + Y_A(\boldsymbol{\xi}_i))$ are polynomial in $a, \sqrt{t}, N, \frac{1}{\epsilon}, \log\frac{1}{\delta}$. The length of the storage of each entry is logarithmic in the above quantities.

Finally observe that we need $O(N^3)$ to compute $\det(\sqrt{t}I_N+Y_A(\boldsymbol{\xi}_i))$. Hence the total number of computations for our estimate is of order

$$t^{-1}a^2N^4\epsilon^{-2}\log\delta^{-1}.$$

□

The quantity $\frac{1}{N}\log\Phi(t,G_\omega)$ can be viewed as the *exponential growth* of $\log\Phi(t,G_\omega)$ in terms of the number of vertices N of G. Note that since the total number of matching of a graph G is given by $\Phi(1,G_\iota)$, Theorem 4.1 combined with (1.7) yields that the exponential growth of the computable lower bound $\tilde\Phi(1,G_\iota)$ differs by $\frac{1}{2}$ at most from the exponential growth of $\Phi(1,G_\iota)$. Note that for complete graphs on $2n$, the exponential growth of the number of perfect matching matchings is of order $\log 2n-1$. For k-regular bipartite graphs on $2n$ vertices the results of [4, 7] imply the inequality that for n big enough the exponential growth of the total number of matchings is at least $\log k-1$. Thus for graphs G on $2n$ vertices containing, bipartite k-regular graphs on $2n$ vertices, with $k\ge 5$ and n big enough, $\tilde\Phi(1,G_\iota)$ has a positive exponential growth.

5 Another estimate of $\log\Phi(t,G_\omega)-\log\tilde\Phi(t,G_\omega)$

Lemma 5.1 *Let X be a real Gaussian random variable. Then*

$$\log \mathrm{E}\, X^2 - \mathrm{E}\, \log X^2 \le C_1, \tag{5.1}$$

where C_1 is given by (1.1). Equality holds if and only if $\mathrm{E}\, X = 0$.

Proof. Clearly, it is enough to prove the lemma in the case $X = Y + a$, where Y is a normalized by (1.2) and $a\ge 0$. In that case the left-hand side of (5.1) is equal to

$$g(a) := \log(1+a^2) - \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\log((x+a)^2)e^{-\frac{x^2}{2}}dx.$$

We used the software Maple to show that $f(a)$ is a decreasing function on $[0,\infty)$. So $f(0)=C_1$ and $\lim_{a\to\infty} f(a) = 0$. This proves the inequality (5.1).

Equality holds if and only if $X = bY$ for some $b \neq 0$. $\square$

Denote by $\mathrm{S_n} \subset \mathbb{R}^{\mathrm{n\times n}}$ the space of $n \times n$ real symmetric matrices. A polynomial $P : \mathbb{R}^n \to \mathbb{R}$ is of degree 2 if

$$P(\mathbf{x}) = \mathbf{x}^\top Q\mathbf{x} + 2\mathbf{a}^\top \mathbf{x} + b,$$
$$\mathbf{x} = (x_1, \ldots, x_n)^\top, \mathbf{a} = (a_1, \ldots, a_n)^\top \in \mathbb{R}^n, Q \in \mathrm{S_n}, \mathrm{b} \in \mathbb{R}.$$

(We allow here the case $Q = 0$.) The quadratic form $P_h : \mathbb{R}^{n+1} \to \mathbb{R}$ induced by P is given

$$P_h(\mathbf{y}) = \mathbf{y}^\top Q_h \mathbf{y}, Q_h = \left[\begin{array}{cc} Q & \mathbf{a} \\ \mathbf{a}^\top & b \end{array}\right] \in \mathrm{S_{n+1}}, \mathbf{y} = (\mathrm{y_1}, \ldots, \mathrm{y_{n+1}})^\top.$$

Clearly, $P(\mathbf{x}) = P_h((\mathbf{x}^\top, 1)^\top)$. P is called a nonnegative polynomial if $P(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$. It is well known and a straightforward fact that P is nonnegative if and only if Q_h is a nonnegative definite matrix.

The following lemma is a generalization of [1, Thm 4.2, (1)].

Lemma 5.2 *Let $P : \mathbb{R}^n \to \mathbb{R}$ be a nonzero nonnegative quadratic polynomial. Let $X_1, \ldots, X_n$ be n-Gaussian random variables, and denote $\mathbf{X} := (X_1, \ldots, X_n)^\top$. Then*

$$\mathrm{E}\ \log P(\mathbf{X}) \leq \log \mathrm{E}\ P(\mathbf{X}) \leq \mathrm{E}\ \log P(\mathbf{X}) + C_1, \tag{5.2}$$

where C_1 is given by (1.1).

Proof. We may assume without a loss of generality that $\mathrm{E}\ P = 1$. In view of the concavity of log we need to show the right-hand side of (5.2). Since Q_h is nonnegative definite it follows that

$$P(\mathbf{x}) = \sum_{i=1}^{m} \lambda_i (\mathbf{a}_i^\top \mathbf{x} + b_i)^2,\ \mathbf{a}_i \in \mathbb{R}^n, b_i \in \mathbb{R}, \lambda_i > 0, i = 1, \ldots, m,$$
$$\mathrm{E}\ (\mathbf{a}_i^\top \mathbf{X} + b_i)^2 = 1,\ i = 1, \ldots m, \quad \sum_{i=1}^{m} \lambda_i = 1.$$

Note that one can have at most one $\mathbf{a}_i = \mathbf{0}$, and in that case then $b_i^2 = 1$. The concavity of log yields

$$\log P(\mathbf{X}) \geq \sum_{i=1}^{m} \lambda_i \log(\mathbf{a}_i^\top \mathbf{X} + b_i)^2.$$

(We assume that $\log 0 = -\infty$.) Note that if $\mathbf{a}_i \neq 0$ then $\mathbf{a}_i\mathbf{X} + b_i$ is Gaussian. Lemma 5.1 yields $\mathrm{E}\ \log P(\mathbf{X}) \geq -C_1$. $\square$

Theorem 5.3 *Let the assumptions of Theorem 4.1 hold. Then (1.7) holds.*

Proof. In view of (4.3) it is left to show

$$\log \Phi(t, G_\omega) \le \log \tilde{\Phi}(t, G_\omega) + (N-1)C_1. \tag{5.3}$$

Let $A = [a_{ij}]_{i,j=1}^n \in \mathrm{A_N}$. Recall that $\det A = (\mathrm{pfaf}\, A)^2$, where $\mathrm{pfaf}\, A$ is the pfaffian. (So $\mathrm{pfaf}\, A = 0$ if n is odd.) Let $\mathbf{a}_i = (a_{1i}, \ldots, a_{(i-1)i})^\top \in \mathbb{R}^{i-1}, i = 2, \ldots, n$. We view $\mathrm{pfaf}\, A$ as multilinear polynomial $\mathrm{Pf}(\mathbf{a}_2, \ldots, \mathbf{a}_n)$ of total degree $\frac{n}{2}$, which is linear in each vector variable $\mathbf{a}_i$. (Any polynomial of noninteger total degree is zero polynomial by definition.)

Denote by $Q_{k,n}$ the set of subsets of $\langle n \rangle$ of cardinality $k \in [1, n]$. Each $\alpha \in Q_{k,n}$ is viewed as $\alpha = \{i_1, \ldots, i_k\}, 1 \le i_1 < \ldots < i_k \le m$. For any matrix $B = [b_{ij}] \in \mathbb{R}^{n \times n}$ and $\alpha \in Q_{k,n}$ we define $B[\alpha|\alpha] \in \mathbb{R}^{k \times k}$ as the principal submatrix $[b_{\alpha_i \alpha_j}]_{i,j=1}^k$. Then for $A = [a_{ij}] \in \mathrm{A_n}$ denote

$$\mathrm{Pf}_\alpha(\mathbf{a}_2, \ldots, \mathbf{a}_n) := \mathrm{pfaf}\, A[\alpha|\alpha].$$

Then $\mathrm{Pf}_\alpha(\mathbf{a}_2, \ldots, \mathbf{a}_n)$ is a multilinear polynomial of total degree $\frac{k}{2}$, which is linear in each $\mathbf{a}_i$. Hence

$$\det(sI_N + A) = s^N + \sum_{k=1}^n s^{N-k} \sum_{\alpha \in Q_{k,n}} \mathrm{Pf}_\alpha(\mathbf{a}_2, \ldots, \mathbf{a}_N)^2, \text{ for any } A \in \mathrm{A_N}. \tag{5.4}$$

View $\mathbf{a}_i \in \mathbb{R}^{i-1}$ as a variable while all other $\mathbf{a}_2, \ldots, \mathbf{a}_N$ are fixed. Then for $s \ge 0$ the above polynomial is quadratic and nonnegative. Group the $\binom{N}{2}$ independent normalized random Gaussian variables $X_{ij}, 1 \le i < j \le N$ into $N-1$ random vectors $\mathbf{X}_i := (X_{1i}, \ldots, X_{(i-1)i})^\top, i = 2, \ldots, N$. Consider now Y_A. Let

$$P(\mathbf{X}_2, \ldots, \mathbf{X}_N) := \det(\sqrt{t} I_N + Y_A) \quad t \ge 0.$$

Then $P(\mathbf{X}_2, \ldots, \mathbf{X}_N)$ is a nonnegative quadratic polynomial in each $\mathbf{X}_j, j = 2, \ldots, N$. Denote by $\mathrm{E}_{\ i}$ the expectation with respect to the variables $X_{1i}, \ldots, X_{(i-1)i}$. (5.4) yields that

$$P_i(\mathbf{X}_2, \ldots, \mathbf{X}_i) := \mathrm{E}_{\ i+1} \ldots \mathrm{E}_{\ \mathrm{N}} P(\mathbf{X}_2, \ldots, \mathbf{X}_N)$$

is a nonnegative quadratic polynomial in each $\mathbf{X}_j, j = 2, \ldots, i$. Lemma 5.2 yields

$$\log \mathrm{E}_{\ i} P_i(\mathbf{X}_2, \ldots, \mathbf{X}_i) \le \mathrm{E}_{\ \mathrm{i}} \log P_i(\mathbf{X}_2, \ldots, \mathbf{X}_i) + C_1, \quad i = 2, \ldots, N.$$

Hence

$$\begin{aligned} \log \Phi(t, G_\omega) = \log \mathrm{E}_{\ 2} P_2(\mathbf{X}_2) \le \mathrm{E}_{\ 2} \log P_2(\mathbf{X}_2) + C_1 \le \\ \mathrm{E}_{\ 2} \mathrm{E}_{\ 3} \log P_3(\mathbf{X}_2, \mathbf{X}_3) + 2C_1 \le \ldots \le \\ \mathrm{E}_{\ 2} \mathrm{E}_{\ 3} \ldots \mathrm{E}_{\ N} \log P(\mathbf{X}_2, \mathbf{X}_3, \ldots, \mathbf{X}_N) + (N-1)C_1 = \\ \log \tilde{\Phi}(t, G_\omega) + (N-1)C_1. \end{aligned}$$

$\square$

6 Bipartite graphs

Assume that $G = (V, E)$ is a bipartite graph. So $V = V_1 \cup V_2, E \subset E_1 \times E_2$ and $N = m + n$. Assume for convenience of notation that $m : \#V_1 \le n := \#V_2$. Thus $E \subset \langle m \rangle \times \langle n \rangle$, so each $e \in E$ is identified uniquely with $(i, j) \in \langle m \rangle \times \langle n \rangle$. Let $C = [c_{ij}] \in \mathbb{R}^{m \times n}$ be the weight matrix associated with the weights $\omega : E \to (0, \infty)$. So $c_{ij} = 0$ if $(i, j) \notin E$ and $c_{ij} = \sqrt{\omega(i, j)}$ if $(i, j) \in E$. Let $x_{ij}, i = 1, \ldots, m, j = 1, \ldots, n$ be mn independent normalized real Gaussian variables. Let $U_C =: [c_{ij} x_{ij}] \in \mathbb{R}^{m \times n}$ be a random matrix. Then the skew symmetric matrix A associated with G_ω is given by and the corresponding random matrices Y_A, X_A are given as

$$A = \begin{bmatrix} 0 & C \\ -C^\top & 0 \end{bmatrix}, Y_A = \begin{bmatrix} 0 & U_C \\ -U_C^\top & 0 \end{bmatrix}, X_A = \frac{1}{\sqrt{m+n}} Y_A. \tag{6.1}$$

Denote by

$$\sigma_1(U_C) \ge \ldots \ge \sigma_m(U_C) \ge 0 \tag{6.2}$$

be the first m singular values of U_C. Then the eigenvalues of Y_A consists of $n - m$ zero eigenvalues and the following $2m$ eigenvalues:

$$\pm \imath \sigma_1(U_C), \ldots, \pm \imath \sigma_m(U_C). \tag{6.3}$$

Hence

$$\det(\sqrt{t} I_{m+n} + Y_A) = t^{\frac{n-m}{2}} \prod_{i=1}^{m} (t + \sigma_i(U_C)^2). \tag{6.4}$$

In [9] the authors considered the random matrix $V_C := U_C U_C^\top \in \mathbb{R}^{m \times m}$. Note that the eigenvalues of V_C are

$$\sigma_1^2(U_C) \ge \ldots \ge \sigma_m^2(U_C). \tag{6.5}$$

Furthermore, one has the equality $\mathrm{E} \det V_C = \phi(m, G_\omega)$. Let $K_{m,n}$ be the complete bipartite graph on $V_1 = \langle m \rangle, V_2 = \langle n \rangle$ vertices. Assume that $1 \le m \le n$. Let $0 < b \le a$ be fixed. Denote by $\Omega_{m,n,[b^2,a^2]}$ the sets of all weights $\omega : \langle m \rangle \times \langle n \rangle \to [b^2, a^2]$. Recall that each $\omega \in \Omega_{m,n,[b^2,a^2]}$ induces the positive matrix $C(\omega) = [c_{ij}(\omega)] \in \mathbb{R}^{m \times n}$, where $c_{ij}(\omega) \in [b, a]$. It was shown in [9] that $\frac{1}{n} \log \det V_{C(\omega)}$ concentrates at $\frac{1}{n} \log \phi(m, K_{m,n,\omega})$ with probability 1 as $n \to \infty$. More precisely

$$\limsup_{n \to \infty} \sup_{m \le n, \omega \in \Omega_{m,n,[b^2,a^2]}} \Pr(\frac{1}{n} |\log \det V_{C(\omega)} - \log \phi(m, K_{m,n,\omega})| > \delta) = 0 \tag{6.6}$$

for any $\delta > 0$.

Theorem 6.1 *Let $0 < b \le a$ be given. For $\omega \in \Omega_{m,n,[b^2,a^2]}$ let $C(\omega)$ be a positive $m \times n$ matrix defined above and $A(\omega) \in \mathrm{A_{m+n}}$ be given by (6.1), $(C = C(\omega))$. Assume that $x_{ij}, 1 \le i \le j \le (m + n)$ are independent Gaussian*

satisfying (1.2). Let $Y_A \in \mathrm{A_N}$ be the random skew symmetric matrix given by (1.3). Then for any $t > 0$

$$\limsup_{n\to\infty} \sup_{m\le n,\omega\in\Omega_{m,n,[b^2,a^2]}} \Pr\Big(\frac{1}{m+n}|\log\det(\sqrt{t}I_N+Y_A)-\log\Phi(t,K_{m,n,\omega})|>\delta\Big)=0 \tag{6.7}$$

Equivalently

$$\limsup_{n\to\infty} \sup_{m\le n,\omega\in\Omega_{m,n,[b^2,a^2]}} \frac{1}{m+n}(\log\Phi(t,K_{m,n,\omega})-\log\tilde{\Phi}(t,K_{m,n,\omega}))=0. \tag{6.8}$$

Proof. Our proof follows the arguments in [9], and we point out the modifications that one has to make. Let $N = m+n$. Since $1 \le m \le n$ we have that $\frac{1}{2n} \le \frac{1}{N} < \frac{1}{n}$. (4.2) with $k = 1$ implies:

$$\limsup_{n\to\infty} \sup_{m\le n,\omega\in\Omega_{m,n,[b^2,a^2]}} \Pr\Big(\frac{1}{m+n}|\log\det(\sqrt{t}I_N+Y_A)-\log\tilde{\Phi}(t,K_{m,n,\omega})|>\delta\Big)=0 \tag{6.9}$$

Thus it is enough to show equality (6.8).

Denote by X_A the random hermitian matrix $X_A := \frac{1}{\sqrt{N}}\imath Y_A$. For $\epsilon > 0$ define

$$\det{}_\epsilon(\sqrt{t}I_N+Y_N) := \prod_{i=1}^{N}\sqrt{t+\max(|\lambda_i(Y_N)|,\sqrt{N}\epsilon)^2},$$

$$\det{}_\epsilon(\frac{\sqrt{t}}{\sqrt{N}}I_N-\imath X_N) := \prod_{i=1}^{N}\sqrt{\frac{t}{N}+\max(|\lambda_i(X_N)|,\epsilon)^2}.$$

Clearly,

$$\det{}_\epsilon(\sqrt{t}I_N+Y_N)=N^{\frac{N}{2}}\det{}_\epsilon(\frac{\sqrt{t}}{\sqrt{N}}I_N-\imath X_N). \tag{6.10}$$

Let $f_{N,t,\epsilon}(x) := \frac{1}{2}\log(\frac{t}{N}+\max(|x|,\epsilon)^2)$. Then

$$|f_{N,t,\epsilon}|_{\mathcal{L}} \le \frac{1}{\epsilon} \text{ for } N \ge \frac{t}{\epsilon^2}.$$

In what follows we assume that $N \ge \frac{t}{\epsilon^2}$. Observe next that

$$\frac{1}{N}\log\det{}_\epsilon(\frac{\sqrt{t}}{\sqrt{N}}I_N-\imath X_N)=\mathrm{tr}_N\, f_{N,t,\epsilon}(X_A).$$

Combine the concentration inequality (3.1) with (6.10) to obtain

$$\Pr\Big(|\frac{1}{N}(\log\det{}_\epsilon(\sqrt{t}I_N+Y_N)-\mathrm{E}\ \log\det{}_\epsilon(\sqrt{t}I_N+Y_N))|\ge r\Big)\le 2e^{-\frac{N^2r^2\epsilon^2}{8a^2}} \tag{6.11}$$

Let

$$\epsilon_N = \frac{1}{(\log N)^2}. \tag{6.12}$$

Note that for a fixed t one has $N \geq \frac{t}{\epsilon_N^2}$ for $N >> 1$. Hence

$$\limsup_{N\to\infty} \Pr(\frac{1}{N}|\log \det_{\epsilon_N}(\sqrt{t}I_N + Y_N) - \mathrm{E}\ \log\det_{\epsilon_N}(\sqrt{t}I_N + Y_N)| \geq \delta) = 0$$

for any $\delta > 0$. As in [9, Prf. of Lemma 2.1] use (6.11) and Lemma 3.2 to deduce that

$$\lim_{N\to\infty} \frac{1}{N}(\log \mathrm{E}\ \det_{\epsilon_N}(\sqrt{t}I_N + Y_N) - \mathrm{E}\ \log\det_{\epsilon_N}(\sqrt{t}I_N + Y_N)) = 0,$$

which is equivalent to

$$\lim_{N\to\infty} \frac{1}{N}(\log \mathrm{E}\ \det_{\epsilon_N}(\frac{\sqrt{t}}{\sqrt{N}}I_N - \imath X_N) - \mathrm{E}\ \log\det_{\epsilon_N}(\frac{\sqrt{t}}{\sqrt{N}}I_N - \imath X_N)) = 0. \tag{6.13}$$

It is left to show that under the assumption of the theorem

$$\lim_{N\to\infty} \frac{1}{N}(\log \mathrm{E}\ \det_{\epsilon_N}(\sqrt{t}I_N + Y_N) - \log \mathrm{E}\ \det(\sqrt{t}I_N + Y_N)) = 0. \tag{6.14}$$

Clearly, the above claim is equivalent to

$$\lim_{N\to\infty} \frac{1}{N}(\log \mathrm{E}\ \det_{\epsilon_N}(\frac{\sqrt{t}}{\sqrt{N}}I_N - \imath X_N) - \log \mathrm{E}\ \det(\frac{\sqrt{t}}{\sqrt{N}}I_N - \imath X_N)) = 0. \tag{6.15}$$

To prove the above equality we use the results of [9]. First observe that X_N has at least $n - m$ eigenvalues which are equal to zero, while the other $2m$ eigenvalues are $\pm\lambda_1(X_N), \ldots, \pm\lambda_m(X_N)$. Furthermore $\lambda_1(X_N)^2, \ldots, \lambda_m^2(X_N)$ are the m eigenvalues of $\frac{1}{N}U_C U_C^\top$, denoted in [9] as $Z(\tilde{A}_{n,m})$. Clearly

$$\det_\epsilon(\frac{\sqrt{t}}{\sqrt{N}}I_N - \imath X_N) = (\frac{\sqrt{t}}{\sqrt{N}})^{n-m}\prod_{i=1}^m(\frac{t}{N} + \max(\lambda_i(X_N)^2, \epsilon)^2) \geq$$
$$\det(\frac{\sqrt{t}}{\sqrt{N}}I_N - \imath X_N) = (\frac{\sqrt{t}}{\sqrt{N}})^{n-m}\prod_{i=1}^m(\frac{t}{N} + \lambda_i(X_N)^2). \tag{6.16}$$

Hence for $\epsilon \leq 1$

$$0 \leq \frac{1}{N}(\log\det_\epsilon(\frac{\sqrt{t}}{\sqrt{N}}I_N - \imath X_N) - \log\det(\frac{\sqrt{t}}{\sqrt{N}}I_N - \imath X_N)) =$$
$$\frac{1}{N}\sum_{\lambda_i(X_N)^2\leq\epsilon^2} \log\frac{\frac{t}{N} + \epsilon^2}{\frac{t}{N} + \lambda_i(X_N)^2} \leq \frac{1}{N}\sum_{\lambda_i(X_N)^2\leq\epsilon^2} \log\frac{\epsilon^2}{\lambda_i(X_N)^2} \leq$$
$$\frac{1}{N}\sum_{\lambda_i(X_N)^2\leq\epsilon^2} \log\frac{1}{\lambda_i(X_N)^2}.$$

[9, (3.2)] is equivalent to

$$\limsup_{n\to\infty} \sup_{m\le n,\omega\in\Omega_{m,n,[b^2,a^2]}} \mathrm{E}\ \frac{1}{m+n} \sum_{\lambda_i(X_{m+n})^2\le\epsilon_{m+n}^2} \log\frac{1}{\lambda_i(X_{m+n})^2} = 0.$$

Hence

$$\lim_{N\to\infty}\frac{1}{N}(\mathrm{E}\ \log\det{}_{\epsilon_N}(\frac{\sqrt{t}}{\sqrt{N}}I_N - \imath X_N) - \mathrm{E}\ \log\det(\frac{\sqrt{t}}{\sqrt{N}}I_N - \imath X_N)) = 0. \quad (6.17)$$

Combine (6.16) with Jensen's inequality to deduce

$$\mathrm{E}\ \log\det(\frac{\sqrt{t}}{\sqrt{N}}I_N - \imath X_N) \le \log \mathrm{E}\ \det(\frac{\sqrt{t}}{\sqrt{N}}I_N - \imath X_N) \le \log \mathrm{E}\ \det{}_{\epsilon}(\frac{\sqrt{t}}{\sqrt{N}}I_N - \imath X_N)$$

Hence

$$\limsup_{N\to\infty}\frac{1}{N}(\log\mathrm{E}\ \det{}_{\epsilon_N}(\frac{\sqrt{t}}{\sqrt{N}}I_N - \imath X_N) - \mathrm{E}\ \log\det(\frac{\sqrt{t}}{\sqrt{N}}I_N - \imath X_N)) \ge$$
$$\limsup_{N\to\infty}\frac{1}{N}(\log\mathrm{E}\ \det{}_{\epsilon_N}(\frac{\sqrt{t}}{\sqrt{N}}I_N - \imath X_N) - \log\mathrm{E}\ \det(\frac{\sqrt{t}}{\sqrt{N}}I_N - \imath X_N)) \ge 0.$$

Use (6.13) and (6.17) to deduce (6.15). □

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